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# On the criteria for linear independence of Nesterenko, Fischler and Zudilin

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**Abstract:** In 1985, Yu. V. Nesterenko produced a criterion for linear independence, which is a variant of Siegel's. While Siegel uses upper bounds on full systems of forms, Nesterenko uses upper and lower bounds on sufficiently dense sequences of individual forms. The proof of Nesterenko's criterion was simplified by F. Amoroso and P. Colmez in 2003. More recently, S. Fischler and W. Zudilin produced a refinement, together with a much simpler proof. This new proof rests on a simple argument which we expand here. We get a new result, which contains Nesterenko's criterion, as well as criteria for algebraic independence.

**Keywords:** Linear independence criterion, Siegel's method, Nesterenko's criterion, criteria for transcendence, criteria for algebraic independence.

**2000 Mathematics Subject Classification:** 11J72, 11J85

## 1 Introduction

In his fundamental paper [9] in 1929, C.L. Siegel introduced the following result.

**Theorem 1.1** (Siegel's linear independence criterion). *Let $\underline{\vartheta} = (\vartheta_1, \dots, \vartheta_m) \in \mathbb{R}^m$. Assume that, for all $\varepsilon > 0$, there exists a complete system of $m+1$ linearly independent linear forms in $m+1$ variables*

$$L_i = L_i(\underline{X}) = \sum_{j=0}^{m} b_{ij} X_j, \quad i = 0, 1, \dots, m, \quad b_{ij} \in \mathbb{Z},$$

*such that*

$$\max_{0 \le i \le m} |L_i(1, \underline{\vartheta})| \le \frac{\varepsilon}{A^{m-1}} \quad \textit{where} \quad A = \max_{0 \le i,j \le m} |b_{ij}|.$$

*Then the numbers $1, \vartheta_1, \dots, \vartheta_m$ are linearly independent over $\mathbb{Q}$.*

This result is discussed in [3] Chap. 2 § 1.4. We reproduce the proof here. As pointed out by Siegel, this argument yields not only linear independence results, but also quantitative refinements (measures of linear independence) which we do not consider here. Studying only qualitative aspects of the subject enables us to simplify the situation.

*Proof.* Let

$$L(\underline{X}) = a_0X_0 + \cdots + a_mX_m, \quad a_j \in \mathbb{Z},$$

be a non–zero linear form with integer coefficients in $m+1$ variables. The aim is to prove, under the assumptions of Theorem 1.1, $L(1,\underline{\vartheta}) \neq 0$. Set

$$H = \max_{0\le j\le m} |a_j|.$$

Let $\varepsilon$ be a positive real number $< 1/(m!\cdot mH)$. Among the forms $\{L_0,\dots,L_m\}$ satisfying the assumptions of Theorem 1.1 for this value of $\varepsilon$, there exist $m$ of them, say $L_{k_1},\dots,L_{k_m}$, which along with $L$ make up a complete system of linearly independent forms. Denote by $\Delta$ the determinant of the coefficient matrix of the system of linear forms $L, L_{k_1},\dots,L_{k_m}$ and, for $0\le i,j\le m$, by $\Delta_{i,j}$ the $(i,j)$-minor of this matrix. Then

$$\Delta = L(1,\underline{\vartheta})\cdot\Delta_{0,0} + \sum_{i=1}^{m} L_{k_i}(1,\underline{\vartheta})\cdot\Delta_{i,0}.$$

Since $\Delta\in\mathbb{Z}$ and $\Delta\neq 0$, we have $|\Delta|\ge 1$. One easily estimates, for $0\le j\le m$,

$$|\Delta_{0,j}| \le m!A^m \quad \text{and} \quad \max_{1\le i\le m} |\Delta_{i,j}| \le m!HA^{m-1}.$$

It follows that

$$\begin{aligned}
(m!)^{-1} &\le |L(1,\underline{\vartheta}))|\cdot A^m + \sum_{i=1}^{m} |L_{k_i}(1,\underline{\vartheta})|\cdot HA^{m-1}\\
&\le |L(1,\underline{\vartheta})|\cdot A^m + \varepsilon\cdot mH\\
&< |L(1,\underline{\vartheta})|\cdot A^m + (m!)^{-1}.
\end{aligned}$$

Thus $L(1,\underline{\vartheta})\neq 0$. □

Siegel used this approach to prove transcendence results (linear independence of $1,\vartheta,\dots,\vartheta^m,\dots$).

In 1985, Yu. V. Nesterenko (see the corollary of the theorem in [7]) introduced a different type of criterion, involving a sequence of linear forms (and not a sequence of complete systems of linear forms); for each of them, he requires not only an upper bound, but also a lower bound.

**Theorem 1.2** (Nesterenko's linear independence criterion). *Let $c_1, c_2, \tau_1, \tau_2$ be positive real numbers and $\sigma(n)$ a non–decreasing positive function such that*

$$\lim_{n\to\infty} \sigma(n) = \infty \quad \text{and} \quad \overline{\lim_{n\to\infty}} \frac{\sigma(n+1)}{\sigma(n)} = 1.$$

*Let $\underline{\vartheta} = (\vartheta_1, \ldots, \vartheta_m) \in \mathbb{R}^m$. Assume that, for all sufficiently large integers $n$, there exists a linear form with integer coefficients in $m+1$ variables*

$$L_n(\underline{X}) = \ell_{0n}X_0 + \ell_{1n}X_1 + \cdots + \ell_{mn}X_m,$$

*which satisfies the conditions*

$$\sum_{i=0}^{m} |\ell_{in}| \leq e^{\sigma(n)} \quad \text{and} \quad c_1 e^{-\tau_1\sigma(n)} \leq |L_n(1, \underline{\vartheta})| \leq c_2 e^{-\tau_2\sigma(n)}.$$

*Then* $\dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}\vartheta_1 + \cdots + \mathbb{Q}\vartheta_m) \geq (1+\tau_1)/(1+\tau_1-\tau_2)$.

The original proof by Nesterenko was rather involved; it has been simplified by F. Amoroso [1] and this simplification was revisited (and translated from Italian to French) by P. Colmez [2]. Recently, S. Fischler and W. Zudilin [5] obtained a refinement of Nesterenko's Criterion. Their proof of this refinement is much easier than the previous proofs. We develop this refinement in the second section, also we show that this result contains Nesterenko's Criterion. Then we deduce criteria for algebraic independence and transcendence criteria in the third and the fourth sections, respectively.

## 2 Main theorem and its proof

Here is our main result.

**Theorem 2.1.** *Let $\underline{\xi} = (\xi_i)_{i\geq 0}$ be a sequence of real numbers with $\xi_0 = 1$, $(r_n)_{n\geq 0}$ a non–decreasing sequence of positive integers, $(Q_n)_{n\geq 0}$, $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ sequences of positive real numbers such that $\lim_{n\to\infty} A_n^{1/r_n} = \infty$ and, for all sufficiently large integers $n$,*

$$Q_nB_n \leq Q_{n+1}B_{n+1}.$$

*For any integer $n \geq 0$, let*

$$L_n(\underline{X}) = \ell_{0n}X_0 + \ell_{1n}X_1 + \cdots + \ell_{r_nn}X_{r_n}$$

*be a linear form with integer coefficients in* $r_n+1$ *variables. Assume that, for any sufficiently large integer* $n$,

$$\sum_{i=0}^{r_n}|\ell_{in}|\le Q_n,\quad 0<|L_n(\underline{\xi})|\le\frac{1}{A_n}\quad\text{and}\quad\frac{|L_{n-1}(\underline{\xi})|}{|L_n(\underline{\xi})|}\le B_n.$$

*Then*

$$A_n\le 2^{r_n+1}(B_nQ_n)^{r_n}$$

*for all sufficiently large integers* $n$.

*Proof.* For $n$ a sufficiently large integer, let $\mathsf{C}_n$ be the convex symmetric compact set in $\mathbb{R}^{1+r_n}$ defined by

$$|x_0|\le\frac{1}{2|L_n(\underline{\xi})|},\quad |x_0\xi_i-x_i|\le|2L_n(\underline{\xi})|^{1/r_n}\quad(1\le i\le r_n).\tag{1}$$

The volume of $\mathsf{C}_n$ is $2^{r_n+1}$. From Minkowski's Convex Body Theorem (Theorem 2B of Chapter II, [8]), there is a non-zero integer point in $\mathsf{C}_n$. We fix such a $\underline{x}(n)=(x_0(n),\dots,x_{r_n}(n))\in\mathbb{Z}^{r_n+1}\setminus\{0\}$. Since $A_n^{1/r_n}\to\infty$ as $n\to\infty$, we have $x_0(n)\ne0$ for $n$ sufficiently large. Indeed, if we had $x_0(n)=0$ with $A_n>2$, we would have $|x_i(n)|\le(2/A_n)^{1/r_n}<1$, and then $x_i(n)=0$, for each $i=1,...,r_n$, which contradicts the choice of $x(n)$.

From the assumptions and the estimate $|2L_n(\underline{\xi})|^{1/r_n}\le(2/A_n)^{1/r_n}$ for $n$ sufficiently large, it follows that the sequence $|2L_n(\underline{\xi})|^{1/r_n}$ tends to zero as $n\to\infty$. Since $r_n\ge1$, $|L_n(\underline{\xi})|$ also tends to zero.

Now if the sequence $|x_0(n)|$ did not tend to infinity, it admits a bounded subsequence, so a constant subsequence, and we would have $x_0(n)=y$, with $y\in\mathbb{Z}$, $y\ne0$, for all $n$ belonging to an infinite subset $A$ of $\mathbb{N}$. For all integers $i\le\sup r_n$, we would have $\lim_{n\in A,n\to\infty}(y\xi_i-x_i(n))=0$, therefore $x_i(n)=y\xi_i$ for $n\in A$ sufficiently large. Thus $y\xi_i\in\mathbb{Z}$ for all $i$, hence $yL_n(\xi)\in\mathbb{Z}$ for all $n$. Since $L_n(\xi)\ne0$, then we would have $|L_n(\xi)|\ge1/|y|$ for all $n$, which contradicts the fact that $|L_n(\underline{\xi})|$ tends to zero. Therefore $\lim_{n\to\infty}|x_0(n)|=\infty$.

We fix $n$ sufficiently large and we write $\underline{x}$ instead of $\underline{x}(n)$. Let $k$ denote the least positive integer such that

$$|x_0|\le\frac{1}{2|L_k(\underline{\xi})|}.$$

Since $|x_0|$ tends to infinity with $n$, it follows that $k$ also tends to infinity with $n$. Moreover we have $k\le n$ and

$$|x_0|>\frac{1}{2|L_{k-1}(\underline{\xi})|}.$$

Now we can write

$$\sum_{i=0}^{r_k} \ell_{ik} x_i = x_0 \sum_{i=0}^{r_k} \ell_{ik}\xi_i + \sum_{i=0}^{r_k} \ell_{ik}(x_i - x_0\xi_i). \tag{2}$$

The term on the left-hand side of (2) is an integer. On the right-hand side, the first term has absolute value equal to $|x_0 L_k(\underline{\xi})|$. Therefore it is bounded above by $1/2$. We now use the following remark (compare with [5]): *if an integer can be written as a sum $x+y$ of two real numbers with $|x| \le 1/2$, then $|y| \ge |x|$.* Hence

$$\left|\sum_{i=0}^{r_k} \ell_{ik}(x_i - x_0\xi_i)\right| \ge \left|x_0 \sum_{i=0}^{r_k} \ell_{ik}\xi_i\right|. \tag{3}$$

The term on the left-hand side of (3) is bounded above by

$$\sum_{i=0}^{r_k} |\ell_{ik}| \max_{1\le i\le r_n} |x_i - x_0\xi_i| \le Q_k (2|L_n(\underline{\xi})|)^{1/r_n} \le Q_k\left(\frac{2}{A_n}\right)^{1/r_n},$$

while the term on the right-hand side of (3) is bounded below by

$$|x_0|\cdot|L_k(\underline{\xi})| \ge \frac{1}{2|L_{k-1}(\underline{\xi})|}\cdot|L_k(\underline{\xi})| \ge \frac{1}{2B_k}\cdot$$

Thus

$$A_n \le 2(2B_kQ_k)^{r_n}.$$

Since $B_kQ_k \le B_nQ_n$, this implies $A_n \le 2(2B_nQ_n)^{r_n}$. □

**Remark 2.2.** Theorem 2.1 does not contain Theorem 6 of [5], since the latter introduces refinements involving gcd's, but its proof relies on the same arguments.

We deduce from Theorem 2.1 a slight refinement of Theorem 1.2 (Nesterenko's linear independence criterion).

**Corollary 2.3.** *Let $\tau_1, \tau_2$ be positive real numbers and $\sigma(n)$ a non–decreasing positive function such that $\lim_{n\to\infty}\sigma(n) = \infty$. Let $\underline{\vartheta} = (\vartheta_1, \dots, \vartheta_m) \in \mathbb{R}^m$. Assume that, for all sufficiently large integers $n$, there exists a linear form with integer coefficients in $m+1$ variables*

$$L_n(\underline{X}) = \ell_{0n}X_0 + \ell_{1n}X_1 + \cdots + \ell_{mn}X_m$$

*which satisfies the conditions*

$$\sum_{i=0}^{m} |\ell_{in}| \le e^{\sigma(n)} \quad \text{and} \quad e^{-(\tau_1+o(1))\sigma(n)} \le |L_n(1,\underline{\vartheta})| \le e^{-(\tau_2+o(1))\sigma(n+1)}.$$

*Then* $\dim_{\mathbb{Q}}(\mathbb{Q} + \mathbb{Q}\vartheta_1 + \cdots + \mathbb{Q}\vartheta_m) \ge (1+\tau_1)/(1+\tau_1-\tau_2)$.

**Remark 2.4.** 1. If $\ell$ is an integer and if the assumptions of Corollary 2.3 are satisfied for parameters $\tau_1$ and $\tau_2$ such that $\tau_1 > (\ell-1)+(\tau_2-\tau_1)\ell$, then the conclusion yields $\dim_{\mathbb{Q}}(\mathbb{Q}+\mathbb{Q}\vartheta_1+\cdots+\mathbb{Q}\vartheta_m) \geq \ell+1$. In particular, if $\tau_1 > (m-1)+m(\tau_1-\tau_2)$, then $1,\vartheta_1,\ldots,\vartheta_m$ are linearly independent over $\mathbb{Q}$.
2. The hypotheses of Corollary 2.3 imply that $L_n(1,\underline{\vartheta}) \neq 0$ and $\lim_{n\to\infty} L_n(1,\underline{\vartheta}) = 0$, therefore $\underline{\vartheta} \notin \mathbb{Q}^m$. When the aim is to prove $\underline{\vartheta} \notin \mathbb{Q}^m$, there is no need of a lower bound for $|L_n(1,\underline{\vartheta})|$. However, assuming a lower bound for $|L_n(1,\underline{\vartheta})|$ enables Nesterenko in [7] to reach a quantitative estimate. Consider for instance the special case $m=1$ with $\vartheta_1=\vartheta\in\mathbb{R}$: under the assumptions of Theorem 1.2, $\vartheta$ is not a Liouville number. Conversely, Theorem 1 of [4] shows that for a real number $\vartheta=\vartheta_1$ which is not a Liouville number, the assumptions of Corollary 2.3 with $m=1$ are satisfied for suitable values of the parameters.

*Proof of Corollary 2.3.* Set $\vartheta_0=1$. Let $\{\xi_0,\xi_1,\ldots,\xi_r\}$ with $\xi_0=1$ be a basis of the vector space spanned by $\vartheta_0,\ldots,\vartheta_m$ over $\mathbb{Q}$. Let $\underline{\xi}=(\xi_1,\ldots,\xi_r)\in\mathbb{R}^r$. Then for $0\leq i\leq m$, there are integers $d,c_{i0},\ldots,c_{ir}$ with $d>0$ such that $d\vartheta_i=\displaystyle\sum_{j=0}^{r} c_{ij}\xi_j$. For any $n\in\mathbb{N}$, define a linear form in $r+1$ variables $Y_0,\ldots,Y_r$ with integer coefficients by

$$\begin{aligned}\Lambda_n(Y_0,\ldots,Y_r) &= L_n\left(\sum_{j=0}^{r} c_{0j}Y_j,\ldots,\sum_{j=0}^{r} c_{mj}Y_j\right)\\ &= \sum_{i=0}^{m} c_{i0}\ell_{in}Y_0+\cdots+\sum_{i=0}^{m} c_{ir}\ell_{in}Y_r.\end{aligned}$$

From the definition of $\Lambda_n$ we infer $\Lambda_n(1,\underline{\xi})=d\cdot L_n(1,\underline{\vartheta})$. We apply Theorem 2.1 with a finite sequence $\xi_0,\xi_1,\ldots,\xi_r$ with $r_n=r$ for all $n$. Let $\tau_1'$ and $\tau_2'$ satisfy $\tau_1'>\tau_1$ and $\tau_2'<\tau_2$. We take

$$Q_n = ce^{\sigma(n)}, \quad A_n = e^{\tau_2'\sigma(n+1)} \quad \text{and} \quad B_n = e^{(\tau_1'-\tau_2')\sigma(n)},$$

where $c>0$ is a suitable constant. We obtain

$$e^{\tau_2'\sigma(n+1)} \leq 2^{r+1}c^r e^{r(\tau_1'-\tau_2'+1)\sigma(n)}.$$

Since $\sigma(n)$ is non–decreasing and tends to infinity, this estimate implies

$$\tau_2' \leq r(1+\tau_1'-\tau_2').$$

Since this last inequality holds for all $(\tau_1',\tau_2')$ with $\tau_1'>\tau_1$ and $\tau_2'<\tau_2$, we deduce $r\geq\tau_2/(1+\tau_1-\tau_2)$. Therefore

$$\dim_{\mathbb{Q}}(\mathbb{Q}+\mathbb{Q}\vartheta_1+\cdots+\mathbb{Q}\vartheta_m) = r+1 \geq \frac{\tau_2}{1+\tau_1-\tau_2}+1 = \frac{1+\tau_1}{1+\tau_1-\tau_2}\cdot$$

□

# 3 Criteria for algebraic independence

We deduce from Theorem 2.1 the following criterion for algebraic independence:

**Corollary 3.1.** *Let $\vartheta_1, \dots, \vartheta_t$ be real numbers, $(d_n)_{n\geq 0}$ a sequence of positive integers, $(\alpha_n)_{n\geq 0}, (\beta_n)_{n\geq 0}$ and $(\gamma_n)_{n\geq 0}$ sequences of positive real numbers such that $\lim_{n\to\infty} d_n^{-t}\gamma_n = \infty$ and, for all sufficiently large integers $n$,*

$$\alpha_n + \beta_n - \gamma_{n-1} \leq \alpha_{n+1} + \beta_{n+1} - \gamma_n.$$

*Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X_1, \dots, X_t]$, where $P_n$ has total degree at most $d_n$ and length at most $e^{\beta_n}$, satisfying*

$$e^{-\alpha_n} \leq |P_n(\vartheta_1, \dots, \vartheta_t)| \leq e^{-\gamma_n}.$$

*Then*

$$\gamma_n \leq \log 2 + \left( \binom{d_n + t}{t} - 1 \right) (\alpha_n + \beta_n - \gamma_{n-1} + \log 2)$$

*for all sufficiently large integers $n$.*

*Proof.* The number of integer tuples $\underline{i} = (i_1, \dots, i_t)$ in $\mathbb{Z}_{\geq 0}^t$ with $i_1 + \cdots + i_t \leq d_n$ is $\binom{d_n + t}{t}$. For each of these $\underline{i}$, let $\xi_{\underline{i}} = \vartheta_1^{i_1} \dots \vartheta_t^{i_t}$. Let $n \in \mathbb{N}$. We write

$$P_n(X_1, \dots, X_t) = \sum_{\underline{i}} a_{\underline{i},n} X_1^{i_1} \dots X_t^{i_t}.$$

Put $L_n(\underline{X}) = \sum_{\underline{i}} a_{\underline{i},n} X_{\underline{i}}$. Then $L_n(\underline{\xi}) = P_n(\vartheta_1, \dots, \vartheta_t)$. Applying Theorem 2.1 with $r_n = \binom{d_n + t}{t} - 1$, $Q_n = e^{\beta_n}$, $A_n = e^{\gamma_n}$ and $B_n = e^{\alpha_n - \gamma_{n-1}}$, we obtain

$$\gamma_n \leq \log 2 + \left( \binom{d_n + t}{t} - 1 \right) (\alpha_n + \beta_n - \gamma_{n-1} + \log 2)$$

for all sufficiently large integers $n$. □

We deduce from Corollary 3.1 the following special cases. In the first one, we consider a sequence of bounded degree polynomials. Since our method relies on linear elimination, our results are sharp when the dimension of the space is not too large. As far as criteria for algebraic independence are concerned, we get sharper results when the degree is bounded.

**Corollary 3.2.** *Let $\vartheta_1, \ldots, \vartheta_t$ be real numbers, $d$ a positive integer, $\tau_1, \tau_2$ positive real numbers and $\sigma(n)$ a non–decreasing positive function such that $\lim_{n\to\infty} \sigma(n) = \infty$. Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X_1, \ldots, X_t]$, where $P_n$ has total degree at most $d$ and length at most $e^{\sigma(n)}$, satisfying*

$$e^{-(\tau_1+o(1))\sigma(n)} \leq |P_n(\vartheta_1, \ldots, \vartheta_t)| \leq e^{-(\tau_2+o(1))\sigma(n+1)}.$$

*Then*

$$\tau_2 \leq \left(\binom{d+t}{t} - 1\right)(1 + \tau_1 - \tau_2).$$

*Proof.* Let $\varepsilon$ be a positive real number $< \tau_2$. Set $\tau_1' = \tau_1 + \varepsilon$ and $\tau_2' = \tau_2 - \varepsilon$. We apply Corollary 3.1 with $d_n = d$, $\alpha_n = \tau_1'\sigma(n)$, $\beta_n = \sigma(n)$ and $\gamma_n = \tau_2'\sigma(n+1)$. The conclusion follows by letting $\varepsilon$ tend to $0$. $\square$

Here are examples with sequences of polynomials whose degrees could tend to infinity.

**Corollary 3.3.** *Let $\vartheta_1, \ldots, \vartheta_t$ be real numbers, $\beta$, $\delta$, $\kappa$ and $\lambda$ positive real numbers, $\alpha$ and $\gamma$ real numbers. Assume $\kappa \leq 1/t$. Further, assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X_1, \ldots, X_t]$, where $P_n$ has total degree at most $\delta n^\kappa + o(n^\kappa)$ and length at most $e^{\beta n + o(n)}$, satisfying*

$$e^{-\lambda n^{1+t\kappa} - (\alpha+o(1))n} \leq |P_n(\vartheta_1, \ldots, \vartheta_t)| \leq e^{-\lambda n^{1+t\kappa} - (\gamma+o(1))n}.$$

*Then*

$$\lambda(1 - 2\delta^t/t!) \leq \delta^t(\alpha + \beta - \gamma)/t! \quad \textit{when} \quad \kappa = 1/t,$$

*and*

$$\lambda \leq \delta^t(\alpha + \beta - \gamma)/t! \quad \textit{when} \quad \kappa < 1/t.$$

*Proof.* Let $\varepsilon$ be a positive real number. Set $\alpha' = \alpha + \varepsilon$, $\beta' = \beta + \varepsilon$, $\gamma' = \gamma - \varepsilon$ and $\delta' = \delta + \varepsilon$. Let $d_n = \delta' n^\kappa$, $\beta_n = \beta' n$, $\gamma_n = \lambda n^{1+t\kappa} + \gamma' n$,

$$\alpha_n = \begin{cases} \lambda n^2 + \alpha' n & \text{when} \quad \kappa = 1/t, \\ \lambda(n-1)^{1+t\kappa} + \alpha' n & \text{when} \quad \kappa < 1/t. \end{cases}$$

We obtain $e^{-\alpha_n} \leq |P_n(\vartheta_1, \ldots, \vartheta_t)| \leq e^{-\gamma_n}$ for $n$ sufficiently large. Besides,

$$\alpha_n + \beta_n - \gamma_{n-1} = \begin{cases} (2\lambda + \alpha' + \beta' - \gamma')n - \lambda + \gamma' & \text{when} \quad \kappa = 1/t, \\ (\alpha' + \beta' - \gamma')n + \gamma' & \text{when} \quad \kappa < 1/t. \end{cases}$$

Since the hypothesis implies that $\alpha-\gamma\geq 0$, it follows that the sequence

$$(\alpha_n+\beta_n-\gamma_{n-1})_{n\geq 1}$$

is non–decreasing for sufficiently large $n$. The conclusion follows by applying Corollary 3.1 and letting $\varepsilon$ tend to $0$. $\square$

**Corollary 3.4.** *Let $\vartheta_1,\ldots,\vartheta_t$ be real numbers, $(d_n)_{n\geq 0}$ a sequence of positive integers and $\beta$, $\kappa$ and $\lambda$ positive real numbers. Assume*

$$d_n=o(n^{1/t}) \quad \textit{and} \quad d_n^t-d_{n-1}^t\leq \kappa$$

*for all sufficiently large integers $n$. Further, assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X_1,\ldots,X_t]$, where $P_n$ has total degree at most $d_n$ and length at most $e^{\beta n+o(n)}$ satisfying*

$$|P_n(\vartheta_1,\ldots,\vartheta_t)|=e^{-(\lambda n d_n^t+o(n))}.$$

*Then*

$$\lambda(t!-\kappa)\leq\beta.$$

*Proof.* Let $\varepsilon>0$, $\alpha_n=\lambda(n-1)(d_{n-1}^t+\kappa)+\varepsilon n$, $\gamma_n=\lambda n d_n^t-\varepsilon n$ and $\beta_n=\beta n+\varepsilon n$. For $n$ sufficiently large, since

$$\lambda n(d_n^t-d_{n-1}^t)+\lambda(d_{n-1}^t+\kappa)+o(n)\leq\lambda n\kappa+o(n)\leq\lambda n\kappa+\varepsilon n,$$

we have

$$\lambda n d_n^t+o(n)\leq\lambda(n-1)(d_{n-1}^t+\kappa)+\varepsilon n.$$

It follows that $|P_n(\vartheta_1,\ldots,\vartheta_t)|\geq e^{-\alpha_n}$.

Also, $|P_n(\vartheta_1,\ldots,\vartheta_t)|\leq e^{-\gamma_n}$ for $n$ sufficiently large.

Since $\alpha_n+\beta_n-\gamma_{n-1}=(\lambda\kappa+\beta+3\varepsilon)n-\lambda\kappa-\varepsilon$, it follows that the sequence $(\alpha_n+\beta_n-\gamma_{n-1})_{n\geq 1}$ is non–decreasing for sufficiently large $n$. The conclusion follows by applying Corollary 3.1 and letting $\varepsilon$ tend to $0$. $\square$

Algebraic independence results follow from our criteria:

**Corollary 3.5.** *Let $\vartheta_1,\ldots,\vartheta_t$ be real numbers, $d$, $\delta$ two positive integers with $d\geq\delta$ and $\tau$, $\eta$ two positive real numbers satisfying*

$$\frac{\tau}{1+\eta}>\binom{t+d}{t}-\binom{t+d-\delta}{t}-1.$$

*Let $\big(\sigma(n)\big)_{n\geq 1}$ be a non–decreasing sequence of real numbers which tends to infinity. Assume that there is a sequence $(P_n)_{n\geq n_0}$ of polynomials in $\mathbb{Z}[X_1,\ldots,X_t]$, where $P_n$ has total degree $\leq d$ and length $\leq e^{\sigma(n)}$, such that*

$$e^{-(\tau+\eta+o(1))\sigma(n)} \leq |P_n(\vartheta_1,\ldots,\vartheta_t)| \leq e^{-(\tau+o(1))\sigma(n+1)}.$$

*Then $\vartheta_1,\ldots,\vartheta_t$ do not satisfy any algebraic dependence relation with rational coefficients of degree $\leq \delta$.*

*Proof.* We use Corollary 2.3 with $\tau_1 = \tau+\eta$, $\tau_2 = \tau$, and $1,\vartheta_1,\ldots,\vartheta_m$ in Corollary 2.3 replaced by $\vartheta_1^{i_1}\cdots\vartheta_t^{i_t}$ with $i_1+\cdots+i_t \leq d$. Then the dimension of the subspace of $\mathbb{R}$ over $\mathbb{Q}$ spanned by $\{\vartheta_1^{i_1}\cdots\vartheta_t^{i_t} \mid i_1+\cdots+i_t \leq d\}$ is bounded below by

$$1+\frac{\tau}{1+\eta}\cdot$$

Now, we suppose that there exists a polynomial $Q \in \mathbb{Z}[X_1,\ldots,X_t]$ of total degree $\leq \delta$ such that $Q(\vartheta_1,\ldots,\vartheta_t) = 0$. Let $\psi$ be the linear transformation from the set of all polynomials of total degree $\leq d$ in $\mathbb{Q}[X_1,\ldots,X_t]$ to $\mathbb{R}$ defined by $\psi(P(X_1,\ldots,X_t)) = P(\vartheta_1,\ldots,\vartheta_t)$ for $P(X_1,\ldots,X_t) \in \mathbb{Q}[X_1,\ldots,X_t]$. Then

$$\{X_1^{j_1}\cdots X_t^{j_t}\cdot Q \mid j_1+\cdots+j_t \leq d-\delta\} \subset \ker\psi.$$

It follows that

$$\dim(\ker\psi) \geq \binom{t+d-\delta}{t}.$$

Therefore, the dimension of the image of $\psi$, which is equal to

$$\binom{t+d}{t} - \dim(\ker\psi),$$

is bounded above by

$$\binom{t+d}{t} - \binom{t+d-\delta}{t}.$$

Since the image of $\psi$ is the subspace of $\mathbb{R}$ over $\mathbb{Q}$ spanned by

$$\{\vartheta_1^{i_1}\cdots\vartheta_t^{i_t} \mid i_1+\cdots+i_t \leq d\},$$

we have

$$1+\frac{\tau}{1+\eta} \leq \binom{t+d}{t} - \binom{t+d-\delta}{t},$$

which is a contradiction. ∎

When the assumptions of Corollary 3.5 are satisfied for any $\delta > 0$, we deduce the algebraic independence of the numbers $\vartheta_1,\ldots,\vartheta_t$.

**Corollary 3.6.** *Let $\vartheta_1, \ldots, \vartheta_t$ be real numbers and $(\tau_d)_{d\geq 1}$, $(\eta_d)_{d\geq 1}$ two sequences of positive real numbers satisfying*

$$\frac{\tau_d}{d^{t-1}(1+\eta_d)} \longrightarrow +\infty.$$

*Further, let $\big(\sigma(n)\big)_{n\geq 1}$ be a non–decreasing sequence of real numbers which tends to infinity. Assume that for all sufficiently large $d$, there is a sequence $(P_n)_{n\geq n_0(d)}$ of polynomials in $\mathbb{Z}[X_1, \ldots, X_t]$, where $P_n$ has total degree $\leq d$ and length $\leq e^{\sigma(n)}$, such that, for $n \geq n_0(d)$,*

$$e^{-(\tau_d+\eta_d)\sigma(n)} \leq |P_n(\vartheta_1, \ldots, \vartheta_t)| \leq e^{-\tau_d\sigma(n+1)}.$$

*Then $\vartheta_1, \ldots, \vartheta_t$ are algebraically independent.*

*Proof.* Let $\delta$ be a positive integer. For sufficiently large $d$, we have

$$\binom{t+d}{t} - \binom{t+d-\delta}{t} - 1 < cd^{t-1} < \frac{\tau_d}{1+\eta_d}$$

where $c > 0$ is a suitable constant. Then the hypotheses of Corollary 3.5 are satisfied with $\tau = \tau_d$ and $\eta = \eta_d$. Hence $\vartheta_1, \ldots, \vartheta_t$ do not satisfy any algebraic dependence relation with rational coefficients of degree $\leq \delta$. This is true for all $\delta$. Therefore $\vartheta_1, \ldots, \vartheta_t$ are algebraically independent. □

# 4 Transcendence criteria

The special case $t = 1$ of Corollary 3.1 yields a criterion for transcendence, which is similar to a well known result due to A.O. Gel'fond (see for instance Lemma 6.3, § 1.3, Chap. 6 of [3]).

**Corollary 4.1.** *Let $\vartheta$ be a real number, $(d_n)_{n\geq 0}$ a sequence of positive integers, $(\alpha_n)_{n\geq 0}, (\beta_n)_{n\geq 0}$ and $(\gamma_n)_{n\geq 0}$ sequences of positive real numbers such that* $\lim_{n\to\infty} \gamma_n/d_n = \infty$ *and, for all sufficiently large integers $n$, $\alpha_n + \beta_n - \gamma_{n-1} \leq \alpha_{n+1} + \beta_{n+1} - \gamma_n$. Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X]$, where $P_n$ has degree at most $d_n$ and length at most $e^{\beta_n}$, satisfying*

$$e^{-\alpha_n} \leq |P_n(\vartheta)| \leq e^{-\gamma_n}.$$

*Then*

$$\gamma_n \leq \log 2 + d_n(\alpha_n + \beta_n - \gamma_{n-1} + \log 2)$$

*for all sufficiently large integers $n$.*

Here is the special case $t = 1$ of Corollary 3.2.

**Corollary 4.2.** *Let $\vartheta$ be a real number, $d$ a positive integer, $\tau_1, \tau_2$ positive real numbers and $\sigma(n)$ a non–decreasing positive function such that $\lim_{n\to\infty} \sigma(n) = \infty$. Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X]$, where $P_n$ has degree at most $d$ and length at most $e^{\sigma(n)}$, satisfying*

$$e^{-(\tau_1+o(1))\sigma(n)} \leq |P_n(\vartheta)| \leq e^{-(\tau_2+o(1))\sigma(n+1)}.$$

*Then*

$$\tau_2 \leq d + d(\tau_1 - \tau_2).$$

Note that if $\tau_1 = \tau_2$, then the conclusion is $\tau_1 \leq d$.

It is interesting to compare Corollary 4.2 with the results following from the proof of Gel'fond's criterion: in our present paper, we use only linear elimination, while Gel'fond's proof relies on algebraic elimination. In Gel'fond's criterion, there is no need of a lower bound for $|P_n(\vartheta)|$, but the conclusion is not so sharp. For instance Lemma 14 [6] implies the following result:

**Proposition 4.3** ([6])**.** *Let $\vartheta$ be a complex number, $d$ a positive integer, $\alpha$ and $\beta$ positive real numbers. Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X]$, where $P_n$ has degree at most $d$ and length at most $e^{\beta n+o(n)}$, satisfying $0 < |P_n(\vartheta)| \leq e^{-\alpha n+o(n)}$. Then $\alpha \leq 3d\beta$.*

Under the assumptions of Proposition 4.3, if we assume that $\vartheta$ is a real number and

$$|P_n(\vartheta)| = e^{-\alpha n+o(n)},$$

then Corollary 4.2 gives the stronger conclusion $\alpha \leq d\beta$.

We conclude with the special case $t = \kappa = 1$ of Corollary 3.3, which is also the special case where $\delta_n$, $\alpha_n$, $\beta_n$ and $\gamma_n$ satisfy

$$\delta_n = \delta n + o(n),\ \alpha_n = \lambda n^2 + \alpha n + o(n),\ \beta_n = \beta n + o(n) \ \text{ and } \ \gamma_n = \lambda n^2 + \gamma n + o(n)$$

of Corollary 4.1.

**Corollary 4.4.** *Let $\vartheta$ be a real number, $\beta$, $\delta$ and $\lambda$ positive real numbers such that $\delta < 1/2$, $\alpha$ and $\gamma$ real numbers. Assume that there exists a sequence $(P_n)_{n\geq 0}$ of polynomials in $\mathbb{Z}[X]$, where $P_n$ has degree at most $\delta n + o(n)$ and length at most $e^{\beta n+o(n)}$, satisfying*

$$e^{-(\lambda n^2+\alpha n+o(n))} \leq |P_n(\vartheta)| \leq e^{-(\lambda n^2+\gamma n+o(n))}.$$

*Then $\lambda(1 - 2\delta) \leq \delta(\alpha + \beta - \gamma)$.*

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